Systems of linear equations

Mathematics – FRDIS

Mendel university in Brno

INVESTMENTS IN EDUCATION DEVELOPMENT

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Basic concepts

Definition (System of linear equations)

A system of m linear equations in n unknowns is a collection of equations

$$(*)\qquad \begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

Variables x_1, x_2, ..., x_n are called unknowns. Numbers a_{ij} are called coefficients of the left-hand sides and numbers b_i are called coefficients of the right-hand sides.
A solution of the system is an ordered n-tuple of real numbers $t_1, t_2, \dots t_n$ that make each equation true statement when the values $t_1, t_2, \dots t_n$ are substituted for x_1, x_2, ..., x_n, respectively.
If $b_1 = b_2 = \cdots = b_m = 0$, the system is called homogenous.

Definition (Coefficient matrix, augmented matrix)

- The matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

  is called the coefficient matrix of system $(*)$.

- The matrix

$$\tilde{A} = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

  is called the augmented matrix of system $(*)$.

Matrix notation of $(*)$

Denote

$$\vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

the vector of the right-hand sides and unknowns, respectively. System $(*)$ can be written as the matrix equation

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix},$$

i.e.,

$$A\vec{x} = \vec{b}.$$

Theorem (Frobenius)

System $(*)$ *has a solution if and only if the rank of the coefficient matrix of* $(*)$ *is equal to the rank of the augmented matrix of this system, i.e.,*

$$\operatorname{rank} A = \operatorname{rank} \tilde{A}.$$

Remark

System $(*)$ may have no solution, exactly one solution, or infinitely many solutions.

- If $\operatorname{rank} A < \operatorname{rank} \tilde{A}$, then $(*)$ has no solution.
- If $\operatorname{rank} A = \operatorname{rank} \tilde{A} = n$, then $(*)$ has exactly one solution.
- If $\operatorname{rank} A = \operatorname{rank} \tilde{A} < n$, then $(*)$ has infinitely many solutions. In this case the unknowns can be computed in terms of $n - \operatorname{rank} A$ parameters (free variables).

Homogeneous linear systems have either exactly one solution (namely, $x_1 = 0$, $x_2 = 0$, ..., $x_n = 0$, called the trivial solution) or an infinite number of solutions (including the trivial solution).

Gauss method

1. We convert the augmented matrix $\tilde{A}$ into its row echelon form (using row operations). We find $\operatorname{rank}\tilde{A}$ and $\operatorname{rank}A$ to determine the solvability or nonsolvability of $(*)$(Frobenius theorem).
2. If $\operatorname{rank}A = \operatorname{rank}\tilde{A}$, we rewrite back the row echelon form of $\tilde{A}$ into a system of linear equations (in the original unknowns). This system has the same set of solutions as the original system $(*)$.
3. We solve this new system from below:
    - If $\operatorname{rank}A = \operatorname{rank}\tilde{A}$=n, there is exactly one "new" unknown in each equation of the system. (Other unknowns have been computed from the equations below.) $\Rightarrow$ exactly one solution
    - If $\operatorname{rank}A = \operatorname{rank}\tilde{A} < n$, then there exists at least one equation with $k > 1$ "new" unknowns. In this case, we solve one arbitrary of these unknowns through the other $k-1$ unknowns. These $k-1$ unknowns are called free variables and can be considered as parameters, i.e., they can take any real values $\Rightarrow$ infinitely many solutions. The choice of the free unknowns is not unique, hence the set of solutions can be written in different forms.

Example (One solution)

Solve the system:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 0 \\ 2x_1 + 4x_2 + 7x_3 &= 8 \\ 3x_1 + 5x_2 + 10x_3 &= 10 \end{aligned}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 2 & 0 \\ 2 & 4 & 7 & 8 \\ 3 & 5 & 10 & 10 \end{array}\right) \begin{array}{l} \urcorner^{-2} \;\; \urcorner^{-3} \\ \longleftarrow_{+} \\ \longleftarrow\!\!\longleftarrow_{+} \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & \boxed{2} & 3 & 8 \\ 0 & 2 & 4 & 10 \end{array}\right) \begin{array}{l} \\ \urcorner^{-1} \\ \longleftarrow_{+} \end{array} \sim \left(\begin{array}{ccc|c} 1 & 1 & 2 & 0 \\ 0 & 2 & 3 & 8 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Rank of the coefficient natrix (denote A) and of the augmented matrix (denote $\tilde{A}$):

$$\operatorname{rank}(A) = \operatorname{rank}(\tilde{A}) = 3$$

number of variables: $n = 3$

$\Rightarrow$ 1 solution

From the last matrix (solved from below):

$$\boxed{x_3 = 2}$$

$$2x_2 + 3 \cdot 2 = 8 \Rightarrow \boxed{x_2 = 1}$$

$$x_1 + 1 + 2 \cdot 2 = 0 \Rightarrow \boxed{x_1 = -5}$$

Example (Infinitely many solution, 1 parameter)

Solve the system:
$$\begin{aligned} x_1 - 2x_2 + 3x_3 - 4x_4 &= 4 \\ x_2 - x_3 + x_4 &= -3 \\ x_1 + 3x_2 \qquad\quad - 3x_4 &= 1 \\ -7x_2 + 3x_3 + x_4 &= -3 \end{aligned}$$

$$\left(\begin{array}{cccc|c} \boxed{1} & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 1 & 3 & 0 & -3 & 1 \\ 0 & -7 & 3 & 1 & -3 \end{array}\right) \begin{array}{l} -1 \\ \\ + \\ \\ \end{array} \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & \boxed{1} & -1 & 1 & -3 \\ 0 & 5 & -3 & 1 & -3 \\ 0 & -7 & 3 & 1 & -3 \end{array}\right) \begin{array}{l} \\ -5 \quad 7 \\ + \\ \qquad + \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & 2 & -4 & 12 \\ 0 & 0 & -4 & 8 & -24 \end{array}\right) \Big| : 2 \quad \sim \left(\begin{array}{cccc|c} 1 & -2 & 3 & -4 & 4 \\ 0 & 1 & -1 & 1 & -3 \\ 0 & 0 & 1 & -2 & 6 \end{array}\right)$$

$\mathrm{rank}(A) = \mathrm{rank}(\tilde{A}) = 3$
number of variables: $n = 4$
$\Rightarrow \infty$ solutions, 1 parameter

$$x_3 - 2x_4 = 6 : \boxed{x_4 = t,\ t \in \mathbb{R}} \Rightarrow \boxed{x_3 = 6 + 2t}$$

$$x_2 - (6 + 2t) + t = -3 \Rightarrow \boxed{x_2 = 3 + t}$$

$$x_1 - 2(3 + t) + 3(6 + 2t) - 4t = 4 \Rightarrow \boxed{x_1 = -8}$$

Example (Infinitely many solutions, 2 parameters)

Solve the system:

$$\begin{aligned} x_1 + 2x_2 + 4x_3 - 3x_4 &= 0 \\ 3x_1 + 5x_2 + 6x_3 - 4x_4 &= 0 \\ 4x_1 + 5x_2 - 2x_3 + 3x_4 &= 0 \\ 3x_1 + 8x_2 + 24x_3 - 19x_4 &= 0 \end{aligned}$$

$$\left(\begin{array}{cccc|c} \boxed{1} & 2 & 4 & -3 & 0 \\ 3 & 5 & 6 & -4 & 0 \\ 4 & 5 & -2 & 3 & 0 \\ 3 & 8 & 24 & -19 & 0 \end{array}\right) \begin{array}{lll} {}^{-3} & {}^{-4} & {}^{-3} \\ \leftarrow + & & \\ \leftarrow & + & \\ \leftarrow & & + \end{array}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 2 & 4 & -3 & 0 \\ 0 & -1 & -6 & 5 & 0 \\ 0 & -3 & -18 & 15 & 0 \\ 0 & 2 & 12 & -10 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 2 & 4 & -3 & 0 \\ 0 & -1 & -6 & 5 & 0 \end{array}\right)$$

$\operatorname{rank}(A) = \operatorname{rank}(\tilde{A}) = 2$

number of variables: $n = 4$

$\Rightarrow \infty$ solutions, 2 parameters

$$-x_2 - 6x_3 + 5x_4 = 0 : \quad \boxed{x_4 = t, x_3 = s, \ t, s \in \mathbb{R}}$$

$$\Rightarrow \boxed{x_2 = -6s + 5t}$$

$$x_1 + 2(-6s + 5t) + 4s - 3t = 0 \Rightarrow \boxed{x_1 = 8s - 7t}$$

Example (No solution)

Solve the system:
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 + x_2 + 2x_3 &= 1 \\ 4x_1 + 5x_2 + 8x_3 &= 2 \end{aligned}$$

$$\left(\begin{array}{ccc|c} \boxed{1} & 2 & 3 & 1 \\ 2 & 1 & 2 & 1 \\ 4 & 5 & 8 & 2 \end{array}\right) \begin{array}{l} \urcorner^{-2} \quad \urcorner^{-4} \\ \hookleftarrow_{+} \\ \longleftarrow_{+} \end{array} \sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & \boxed{-3} & -4 & -1 \\ 0 & -3 & -4 & -2 \end{array}\right) \begin{array}{l} \\ \urcorner^{-1} \\ \hookleftarrow_{+} \end{array}$$

$$\sim \left(\begin{array}{ccc|c} 1 & 2 & 3 & 1 \\ 0 & -3 & -5 & -1 \\ 0 & 0 & 0 & -1 \end{array}\right)$$

$$\operatorname{rank}(A) = 2, \quad \operatorname{rank}(\tilde{A}) = 3$$

$\operatorname{rank}(A) \neq \operatorname{rank}(\tilde{A}) \Longrightarrow$ the system has no solution.

Systems with regular coefficient matrices

Next we present two methods which can be used for solving the system $A\vec{x} = \vec{b}$ in case when A is regular.

Theorem (Properties of regular matrices)

Let A be an $n \times n$ square matrix. Then the following statements are equivalent:

1. *A is invertible, i.e., A^{-1} exists.*
2. $\det A \neq 0$
3. $\mathrm{rank} A = n.$
4. *The rows (columns) of A are linearly independent.*
5. *System of linear equations $A\vec{x} = \vec{b}$ has a unique solution for any vector $\vec{b}$.*

Theorem (Method of matrix inversion)

Let A be an $n \times n$ matrix and suppose that A is invertible. Then system of equations $A\vec{x} = \vec{b}$ has a unique solution

$$\vec{x} = A^{-1}\vec{b}.$$

Example

Solve the system:

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + 3x_3 &= 2 \\ x_1 + x_2 + \ \ x_3 &= 3 \end{aligned}$$

The coefficient matrix:

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$$

The vector of the right-hand sides:

$$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$$

The inverse matrix of A:

$$A^{-1} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$$

The vector of solutions: $\vec{x} = A^{-1}\vec{b} = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ -2 \end{pmatrix}$

$$\Longrightarrow \boxed{x_1 = 3,\ x_2 = 2,\ x_3 = -2}.$$

Theorem (Cramer's rule)

Let A be an $n \times n$ matrix and suppose that $\det A \neq 0$. Then system of equations $A\vec{x} = \vec{b}$ has a unique solution. Let D be the determinant of A and let D_i be the determinant of the matrix obtained from A by replacing the i-th column by the vector $\vec{b}$. Then

$$x_i = \frac{D_i}{D}, \quad i = 1, \dots, n.$$

Remark

- Cramer's rule is inefficient for hand calculations, except for 2×2 or 3×3 matrices.
- Cramer's rule is important in case when we are interested in one of the unknowns only, since each of the unknowns can be found without calculating any of the other unknowns.

Example

Using Cramer's rule solve the system:

$$\begin{aligned} 3x_1 + 5x_2 &= \ 1 \\ 7x_1 + 2x_2 &= 8. \end{aligned}$$

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$$D = \begin{vmatrix} 3 & 5 \\ 7 & 2 \end{vmatrix} = 6 - 35 = -29$$

$$D_1 = \begin{vmatrix} 1 & 5 \\ 8 & 2 \end{vmatrix} = 2 - 40 = -38 \quad \Longrightarrow \quad x_1 = \frac{D_1}{D} = \frac{38}{29}$$

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$$D_2 = \begin{vmatrix} 3 & 1 \\ 7 & 8 \end{vmatrix} = 24 - 7 = 17 \quad \Longrightarrow \quad x_2 = \frac{D_2}{D} = -\frac{17}{29}$$

Example

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$$D_2 = \begin{vmatrix} 3 & 1 \\ 7 & 8 \end{vmatrix} = 24 - 7 = 17 \quad \Longrightarrow \quad x_2 = \frac{D_2}{D} = -\frac{17}{29}$$

$$\Longrightarrow \quad \vec{x} = \left(\frac{38}{29}, -\frac{17}{29} \right)$$

Using the computer algebra systems

Solve the system using Wolfram Alpha (`http://www.wolframalpha.com/`):

$$\begin{aligned} x_1 + x_2 + 2x_3 &= 1 \\ 2x_1 + x_2 + 3x_3 &= 2 \\ x_1 + x_2 + x_3 &= 3 \end{aligned}$$

Solution:

```
solve x1+x2+2*x3=1,2x1+x2+3x3=2,x1+x2+x3=3
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