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Indefinite integral

Mathematics – FRDIS

MENDELU

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Antiderivative and indefinite integral

Definition (Antiderivative)

Let f and F be functions defined on an open interval I. If

$$F'(x) = f(x) \quad \text{for every} \quad x \in I,$$

then the function F is called an **antiderivative** of f on the interval I.

It follows from the definition: If F is an antiderivative of f, then $F + c$ (where c is any real constant), is also an antiderivative of f. Indeed,

$$[F(x) + c]' = F'(x) + c' = f(x) + 0 = f(x).$$

This means that if there exists an antiderivative of f on the given interval, then there exist infinitely many antiderivatives of f on this interval.

Example

For example, the following functions are antiderivatives of $y = x^3$:

$$y = \frac{x^4}{4}, \quad y = \frac{x^4}{4} + 3, \quad y = \frac{x^4}{4} - 7.$$

Indeed,

$$\left(\frac{x^4}{4}\right)' = x^3, \quad \left(\frac{x^4}{4} + 3\right)' = x^3, \quad \left(\frac{x^4}{4} - 7\right)' = x^3.$$

Evidently any function of the form $y = \frac{x^4}{4} + c$, where $c \in \mathbb{R}$, is an antiderivative of $y = x^3$.

Question: Are the functions of the form $y = \frac{x^4}{4} + c$ the only antiderivatives of $y = x^3$ or can we find another antiderivative?

The following theorem gives answer to this question.

Theorem (Uniqueness of antiderivative)

Let F and G be two antiderivatives of f on an interval I. Then there exists a constant $c \in \mathbb{R}$ such that $G(x) = F(x) + c$ for every $x \in I$.

The previous theorem says that any two antiderivatives of the same function f differ only by an additive constant.

Definition (Indefinite integral)

The set of all antiderivatives of f is denoted $\int f(x)\,\mathrm{d}x$ and it is called the **indefinite integral**. We write

$$\int f(x)\,\mathrm{d}x = F(x) + c,$$

where F is any antiderivative of f and c is any real constant.

The symbol $\int$ is called the **integral sign**, the function f is called **integrand**, the constant c is called the **constant of integration**. The $\mathrm{d}x$ is part of integral notation and indicates the variable involved. To **integrate** f means to find all antiderivatives of f. The function which has an antiderivative (indefinite integral) is called **integrable**.
It holds:

$$\left(\int f(x)\,\mathrm{d}x\right)' = f(x) \quad \text{and} \quad \int (F(x))'\,\mathrm{d}x = F(x) + c$$

Theorem (Sufficient condition for integrability)

Let f be a continuous function on I. Then there exists an antiderivative of f on I.

- There exist functions which are not continuous and have antiderivatives.
- There exist functions which are continuous (hence they have antiderivatives), but the antiderivatives cannot be expressed using the elementary functions. The antiderivatives of these functions, such as

$$\int e^{-x^2}\,\mathrm{d}x, \quad \int \frac{e^x}{x}\,\mathrm{d}x,$$

$$\int \frac{\sin x}{x}\,\mathrm{d}x, \quad \int \frac{\cos x}{x}\,\mathrm{d}x, \quad \int \sin x^2\,\mathrm{d}x, \quad \int \cos x^2\,\mathrm{d}x$$

are called **higher transcendental functions**.

Basic Formulas for integration

$$\int 0\,\mathrm{d}x = c$$

$$\int 1\,\mathrm{d}x = x + c$$

$$\int x^n\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$$

$$\int e^x\,\mathrm{d}x = e^x + c$$

$$\int a^x\,\mathrm{d}x = \frac{a^x}{\ln a} + c$$

$$\int \frac{1}{x}\,\mathrm{d}x = \ln|x| + c$$

$$\int \cos x\,\mathrm{d}x = \sin x + c$$

$$\int \sin x\,\mathrm{d}x = -\cos x + c$$

$$\int \frac{1}{\cos^2 x}\,\mathrm{d}x = \mathrm{tg} x + c$$

$$\int \frac{1}{\sin^2 x}\,\mathrm{d}x = -\mathrm{cotg} x + c$$

$$\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x = \arcsin x + c$$

$$\int \frac{1}{x^2+1}\,\mathrm{d}x = \mathrm{arctg} x + c$$

- We usually write $\int \mathrm{d}x$ instead of $\int 1 \,\mathrm{d}x$.
- Sometimes we write $\int \frac{\mathrm{d}x}{f(x)}$ instead of $\int \frac{1}{f(x)} \,\mathrm{d}x$.

Basic rules for integration

Theorem (Basic rules for integration)

Let f and g be functions, which are integrable on an interval I a let $c \in \mathbb{R}$. Then the following formulas holds on I :

$$\begin{aligned} \int [f(x) \pm g(x)] \,\mathrm{d}x &= \int f(x) \,\mathrm{d}x \pm \int g(x) \,\mathrm{d}x \\ \int c \cdot f(x) \,\mathrm{d}x &= c \int f(x) \,\mathrm{d}x \end{aligned}$$

We don't have any general rule for integration products and quotients of functions and for the composite function !!!

Example (Integration of power functions and polynomials)

1. $\displaystyle\int x^3\,\mathrm{d}x = \frac{x^4}{4} + c$

2. $\displaystyle\int \sqrt{x}\,\mathrm{d}x = \int x^{\frac{1}{2}}\,\mathrm{d}x = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} = \frac{2}{3}\sqrt{x^3} + c$

3. $\displaystyle\int \frac{1}{x^4}\,\mathrm{d}x = \int x^{-4}\,\mathrm{d}x = \frac{x^{-3}}{-3} = -\frac{1}{3x^3} + c$

4. $\displaystyle\int (x^4 + 2x^3 + x - 2)\,\mathrm{d}x = \frac{x^5}{5} + \frac{x^4}{2} + \frac{x^2}{2} - 2x + c$

Example (Basic formulas and simplification of integrand)

1. Multiplication of the power functions

$$\int x^2\sqrt{x}\,\mathrm{d}x = \int x^{\frac{5}{2}}\,\mathrm{d}x = \frac{x^{\frac{7}{2}}}{\frac{7}{2}} = \frac{2}{7}x^3\sqrt{x} + c$$

2. Decomposition of the numerator of a fraction

$$\begin{aligned}\int \frac{x+3}{x^2}\,\mathrm{d}x &= \int \left(\frac{x}{x^2} + \frac{3}{x^2}\right)\,\mathrm{d}x = \int \left(\frac{1}{x} + 3x^{-2}\right)\,\mathrm{d}x \\ &= \ln|x| + 3\cdot\frac{x^{-1}}{-1} = \ln|x| - \frac{3}{x} + c\end{aligned}$$

3. Decomposition of the improper rational function into a sum of a polynomial and the proper rational function. (We divide the numerator by the denominator.)

$$\int \frac{x^4}{x^2+1}\,\mathrm{d}x = \int \left(x^2 - 1 + \frac{1}{x^2+1}\right)\,\mathrm{d}x = \frac{x^3}{3} - x + \mathrm{arctg}x + c$$

Example (Using formulas for trigonometric functions)

We use mainly the following formulas:

$$\boxed{\mathrm{tg}x = \frac{\sin x}{\cos x}, \quad \mathrm{cotg}x = \frac{\cos x}{\sin x}, \quad \sin^2 x + \cos^2 x = 1}$$

$$\begin{aligned}\int \mathrm{tg}^2 x \, \mathrm{d}x &= \int \frac{\sin^2 x}{\cos^2 x} \, \mathrm{d}x = \int \frac{1-\cos^2 x}{\cos^2 x} \, \mathrm{d}x = \int \left(\frac{1}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x} \right) \mathrm{d}x \\ &= \int \left(\frac{1}{\cos^2 x} - 1 \right) \mathrm{d}x = \mathrm{tg}x - x + c\end{aligned}$$

Method of substitution

Theorem (The 1 st method of substitution, $t = \varphi(x)$)

Let $f(t)$ be a function which is continuous on I and $\varphi(x)$ be a function having continuous derivative on J. Next suppose that $\varphi(x) \in I$ for every $x \in J$. Then the function $f[\varphi(x)]\varphi'(x)$ is continuous on J and it holds on this interval:

$$\int f[\varphi(x)]\varphi'(x) \, \mathrm{d}x = \int f(t) \, \mathrm{d}t,$$

where we substitute $t = \varphi(x)$ on the right-hand side.

- The method can be used if the integrand is of the form "**composite function** $f[\varphi(x)]$ times **the derivative of the interior function** $\varphi'(x)$"
- We write a new variable t instead of the interior function $\varphi(x)$ and we write $\mathrm{d}t$ instead of $\varphi'(x) \, \mathrm{d}x$.
- Let F be an antiderivative of f, then we proceed as follows:

$$\boxed{\int f[\varphi(x)]\varphi'(x) \, \mathrm{d}x = \left| \begin{array}{c} t = \varphi(x) \\ \mathrm{d}t = \varphi'(x) \, \mathrm{d}x \end{array} \right| = \int f(t) \, \mathrm{d}t = F(t) + c = F[\varphi(x)] + c}$$

Example (The 1 st method of substitution)

1. $$\int \sin(3x+2)\,\mathrm{d}x = \left| \begin{matrix} t = 3x+2 \\ \mathrm{d}t = 3\,\mathrm{d}x \\ \mathrm{d}x = \frac{1}{3}\,\mathrm{d}t \end{matrix} \right| = \frac{1}{3}\int \sin t\,\mathrm{d}t = -\frac{1}{3}\cos t + c$$
$$= -\frac{1}{3}\cos(3x+2) + c$$

2. $$\int \sin^2 x \cos x\,\mathrm{d}x = \int (\sin x)^2 \cdot \cos x\,\mathrm{d}x = \left| \begin{matrix} t = \sin x \\ \mathrm{d}t = \cos x\,\mathrm{d}x \end{matrix} \right| = \int t^2\,\mathrm{d}t$$
$$= \frac{t^3}{3} + c = \frac{\sin^3 x}{3} + c$$

3. $$\int x \cdot e^{1-x^2}\,\mathrm{d}x = \left| \begin{matrix} t = 1-x^2 \\ \mathrm{d}t = -2x\,\mathrm{d}x \\ x\,\mathrm{d}x = -\frac{1}{2}\,\mathrm{d}t \end{matrix} \right| = -\frac{1}{2}\int e^t\,\mathrm{d}t = -\frac{1}{2}e^t + c = -\frac{1}{2}e^{1-x^2} + c$$

4. $$\int x^3 \cdot \sqrt{x^4+1}\,\mathrm{d}x = \left| \begin{matrix} t = x^4+1 \\ \mathrm{d}t = 4x^3\,\mathrm{d}x \\ x^3\,\mathrm{d}x = \frac{1}{4}\,\mathrm{d}t \end{matrix} \right| = \frac{1}{4}\int \sqrt{t}\,\mathrm{d}t = \frac{1}{4}\int t^{\frac{1}{2}}\,\mathrm{d}t$$
$$= \frac{1}{4}\cdot\frac{2}{3}t^{\frac{3}{2}} + c = \frac{1}{6}\sqrt{t^3} + c = \frac{1}{6}\sqrt{(x^4+1)^3} + c$$

Composite function with linear interior function

Let F be an antiderivative of f on an interval I. Then for $ax+b \in I$ we have:

$$\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}F(ax+b) + c.$$

Example

$$\int \sin(5x+1)\,\mathrm{d}x$$

- Using substitution:
$$\int \sin(5x+1)\,\mathrm{d}x = \left| \begin{matrix} t = 5x+1 \\ \mathrm{d}t = 5\,\mathrm{d}x \\ \mathrm{d}x = \frac{1}{5}\,\mathrm{d}t \end{matrix} \right| = \frac{1}{5}\int \sin t\,\mathrm{d}t = -\frac{1}{5}\cos t + c =$$
$$-\frac{1}{5}\cos(5x+1) + c$$

- Using the above formula: $ax + b = 5x + 1$
$$f(x) = \sin(x) \implies F(x) = -\cos x$$
$$f(ax+b) = \sin(5x+1) \implies F(ax+b) = -\cos(5x+1)$$
$$\implies \int \sin(5x+1)\,\mathrm{d}x = -\frac{1}{5}\cos(5x+1) + c$$

Example

1. $$\int \cos 2x \, \mathrm{d}x = \frac{1}{2} \sin 2x + c$$
2. $$\int (3-5x)^6 \, \mathrm{d}x = -\frac{1}{5} \cdot \frac{1}{7}(3-5x)^7 = -\frac{1}{35}(3-5x)^7 + c$$
3. $$\int \frac{1}{2x-3} \, \mathrm{d}x = \frac{1}{2} \ln|2x-3| + c$$
4. $$\int e^{2-x} \, \mathrm{d}x = -e^{2-x} + c$$
5. $$\int \sin\frac{x}{2} \, \mathrm{d}x = \int \sin\left(\frac{1}{2}x\right) \, \mathrm{d}x = -2\cos\frac{x}{2} + c$$

The integrals can be solved using the substitutions:

$$t = 2x,\ t = 3-5x,\ t = 2x-3,\ t = 2-x,\ t = \frac{x}{2}.$$

f'/f

$$\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \left| \begin{array}{c} t = f(x) \\ \mathrm{d}t = f'(x)\, \mathrm{d}x \end{array} \right| = \int \frac{1}{t} \, \mathrm{d}t = \ln|t| + c = \ln|f(x)| + c$$

$$\Longrightarrow \boxed{\int \frac{f'(x)}{f(x)} \, \mathrm{d}x = \ln|f(x)| + c}$$

Example

1. $$\int \frac{2x}{x^2-3} \, \mathrm{d}x = \ln\left|x^2-3\right| + c$$
2. $$\int \frac{3x}{x^2-3} \, \mathrm{d}x = \frac{3}{2} \int \frac{2x}{x^2-3} \, \mathrm{d}x = \frac{3}{2} \ln\left|x^2-3\right| + c$$
3. $$\int \mathrm{tg} x \, \mathrm{d}x = \int \frac{\sin x}{\cos x} \, \mathrm{d}x = -\int \frac{-\sin x}{\cos x} \, \mathrm{d}x = -\ln|\cos x| + c$$
4. $$\int \mathrm{cotg} x \, \mathrm{d}x = \int \frac{\cos x}{\sin x} \, \mathrm{d}x = \ln|\sin x| + c$$

Trigonometric substitution

The integral of the the form $\int R(\sin x, \cos x)\,\mathrm{d}x$, where R is a rational function, can be transformed (with using special substitutions) into the integral from a rational function. Special cases:

1. $\int R(\sin x)\cos x\,\mathrm{d}x \Rightarrow t = \sin x$
2. $\int R(\cos x)\sin x\,\mathrm{d}x \Rightarrow t = \cos x$

Remark

- Functions of the type $R(\sin x, \cos x)$: $\dfrac{\sin x\cos^2 x}{\sin^2 x + \cos x}, \ \dfrac{\cos^3 x + 1}{\sin^2 x}$
- Functions of the type $R(\sin x)$: $\dfrac{\sin x + 2}{\sin^2 x}, \ \dfrac{2\sin x}{\sin x - 1}$
- Functions of the type $R(\cos x)$: $\dfrac{\cos x}{\cos^2 x + 3}, \ \dfrac{\cos^2 x + \cos x}{\cos x + 1}$

Example

1. $$\int \frac{\sin x\cos x}{\sin^2 x + 2}\,\mathrm{d}x = \int \frac{\sin x}{\sin^2 x + 2}\cos x\,\mathrm{d}x = \left|\begin{matrix} t = \sin x \\ \mathrm{d}t = \cos x\,\mathrm{d}x \end{matrix}\right| = \int \frac{t}{t^2+2}\,\mathrm{d}t$$
$$= \frac{1}{2}\int \frac{2t}{t^2+2}\,\mathrm{d}t = \frac{1}{2}\ln(t^2+2) + c = \frac{1}{2}\ln(\sin^2 x + 2) + c$$

2. $$\int \frac{\sin x}{\cos^3 x}\,\mathrm{d}x = \int \frac{1}{\cos^3 x}\sin x\,\mathrm{d}x = \left|\begin{matrix} t = \cos x \\ \mathrm{d}t = -\sin x\,\mathrm{d}x \\ \sin x\,\mathrm{d}x = -\,\mathrm{d}t \end{matrix}\right| = -\int \frac{1}{t^3}\,\mathrm{d}t$$
$$= -\int t^{-3}\,\mathrm{d}t = -\frac{t^{-2}}{-2} + c = \frac{1}{2t^2} + c = \frac{1}{2\cos^2 x} + c$$

3. (*) $$\int \frac{\sin^3 x}{\cos x + 3}\,\mathrm{d}x = \int \frac{\sin^2 x}{\cos x + 3}\sin x\,\mathrm{d}x = \int \frac{1-\cos^2 x}{\cos x + 3}\sin x\,\mathrm{d}x$$
$$= \left|\begin{matrix} t = \cos x \\ \mathrm{d}t = -\sin x\,\mathrm{d}x \\ \sin x\,\mathrm{d}x = -\,\mathrm{d}t \end{matrix}\right| = \int \frac{t^2-1}{t+3}\,\mathrm{d}t = \int \left(t - 3 + \frac{8}{t+3}\right)\mathrm{d}t$$
$$= \frac{t^2}{2} - 3t + 8\ln|t+3| + c = \frac{\cos^2 x}{2} - 3\cos x + 8\ln|\cos x + 3| + c$$

Theorem (The 2 nd method of substitution, $x = \varphi(t)$)

Let $f(x)$ be a function which is continuous on I and $\varphi(t)$ be a function having continuous and nonzero derivative on J. Next suppose that $\varphi(J) = I$ Then it holds on I:

$$\int f(x)\,\mathrm{d}x = \int f[\varphi(t)]\varphi'(t)\,\mathrm{d}t,$$

where we substitute $t = \varphi^{-1}(x)$ on the right-hand side and where φ^{-1} is the inverse function to φ.

- We write $\varphi(t)$ instead of x and $\varphi'(t)\,\mathrm{d}t$ instead od $\mathrm{d}x$. The composite function obtained in the integral on the right-hand side seems to be more complicated than the original one, but in some particular cases it can be easier to integrate this composite function.
- We proceed as follows:

$$\boxed{\int f(x)\,\mathrm{d}x = \left|\begin{matrix} x = \varphi(t) \\ \mathrm{d}x = \varphi'(t)\,\mathrm{d}t \end{matrix}\right| = \int f[\varphi(t)]\varphi'(t)\,\mathrm{d}t = F(t) + c = F[\varphi^{-1}(x)] + c},$$

where $F(t)$ is an antiderivative of $f[\varphi(t)]\varphi'(t)$.

Integration of irrational function

We can eliminate roots in the integral

$$\int R(x,\ \sqrt[n_1]{ax+b},\ \sqrt[n_2]{ax+b}, \dots)\,\mathrm{d}x,$$

where R is a rational function. We use the substitution

$$t = \sqrt[s]{ax+b}, \quad \text{where } s \text{ is the least common multiple of } n_1, n_2, \dots.$$

Then:

$$t = \sqrt[s]{ax+b} \Longrightarrow ax + b = t^s$$

$$x = \frac{1}{a}t^s - \frac{b}{a}$$

$$\mathrm{d}x = \frac{1}{a}st^{s-1}\,\mathrm{d}t$$

Example

1. $$\int \frac{x}{\sqrt{x+1}}\,\mathrm{d}x = \left| \begin{matrix} t = \sqrt{x+1} \Rightarrow x = t^2 - 1 \\ \mathrm{d}x = 2t\,\mathrm{d}t \end{matrix} \right| = \int \frac{t^2-1}{t} 2t\,\mathrm{d}t$$
$$= 2\int (t^2-1)\,\mathrm{d}t = 2\left(\frac{t^3}{3} - t\right) + c = \frac{2}{3}(\sqrt{x+1})^3 - 2\sqrt{x+1} + c$$

2. $$\int \frac{2}{\sqrt{x}(x+1)}\,\mathrm{d}x = \left| \begin{matrix} t = \sqrt{x} \Rightarrow x = t^2 \\ \mathrm{d}x = 2t\,\mathrm{d}t \end{matrix} \right| = \int \frac{2}{t(t^2+1)} 2t\,\mathrm{d}t$$
$$= 4\int \frac{1}{t^2+1}\,\mathrm{d}t = 4\,\mathrm{arctg} t + c = 4\,\mathrm{arctg}\sqrt{x} + c$$

3. $$\int \frac{\sqrt{x}}{\sqrt[3]{x}+1}\,\mathrm{d}x = \left| \begin{matrix} t = \sqrt[6]{x} \Rightarrow x = t^6 \\ \mathrm{d}x = 6t^5\,\mathrm{d}t \end{matrix} \right| = \int \frac{t^3}{t^2+1} 6t^5\,\mathrm{d}t = 6\int \frac{t^8}{t^2+1}\,\mathrm{d}t$$
$$= 6\int \left(t^6 - t^4 + t^2 - 1 + \frac{1}{t^2+1}\right)\mathrm{d}t = 6\left(\frac{t^7}{7} - \frac{t^5}{5} + \frac{t^3}{3} - t + \mathrm{arctg} t\right) + c$$
$$= \frac{6\sqrt[6]{x^7}}{7} - \frac{6\sqrt[6]{x^5}}{5} + 2\sqrt{x} - 6\sqrt[6]{x} + 6\cdot\mathrm{arctg}\sqrt[6]{x} + c$$

Integration by parts

Theorem (Integration by parts formula)

Let u and v be function having continuous derivatives on an interval I. Then the following formula holds on I:

$$\int u(x)v'(x)\,\mathrm{d}x = u(x)v(x) - \int u'(x)v(x)\,\mathrm{d}x.$$

1. Integration by parts formula follows from the rule for differentiation of the product of two functions.
2. When using the integration by parts formula we need:
   - to differentiate the function u,
   - to integrate the function v'. This can be a problem!!!

   Moreover, for an effective application of this formula we need the product $u'v$ (which appears in the integral on the right-hand side) to be "simpler in sense of integration" than the original uv'.

Typical integrals for using integration by parts

Let P be a polynomial.

1. $$\int P(x)e^{ax+b}\,\mathrm{d}x, \quad \int P(x)\sin(ax+b)\,\mathrm{d}x, \int P(x)\cos(ax+b)\,\mathrm{d}x$$

   In these cases we differentiate the polynomial and we integrate the exponential (or trigonometric) function.

2. $$\int P(x)\ln^m(ax+b)\,\mathrm{d}x$$
   $$\int P(x)\mathrm{arctg}(ax+b)\,\mathrm{d}x, \quad \int P(x)\mathrm{arccotg}(ax+b)\,\mathrm{d}x$$
   $$\int P(x)\arcsin(ax+b)\,\mathrm{d}x, \quad \int P(x)\arccos(ax+b)\,\mathrm{d}x$$

   In these cases we integrate the polynomial and we differentiate the logarithmic or the cyclometric function. The case $P(x)=1$ is also included.

Example (Integration by parts I)

1. $$\int x\cos x\,\mathrm{d}x = \left|\begin{array}{ll} u=x & v'=\cos x \\ u'=1 & v=\sin x \end{array}\right| = x\sin x - \int \sin x\,\mathrm{d}x$$
   $$= x\sin x + \cos x + c$$

2. $$\int (x^2+1)e^{-x}\,\mathrm{d}x = \left|\begin{array}{ll} u=x^2+1 & v'=e^{-x} \\ u'=2x & v=-e^{-x} \end{array}\right|$$
   $$= (x^2+1)(-e^{-x}) - \int 2x(-e^{-x})\,\mathrm{d}x$$
   $$= -e^{-x}(x^2+1) + 2\int xe^{-x}\,\mathrm{d}x = \left|\begin{array}{ll} u=x & v'=e^{-x} \\ u'=1 & v=-e^{-x} \end{array}\right|$$
   $$= -e^{-x}(x^2+1) + 2\left[x(-e^{-x}) - \int 1\cdot(-e^{-x})\,\mathrm{d}x\right]$$
   $$= -e^{-x}(x^2+1) - 2xe^{-x} + 2\int e^{-x}\,\mathrm{d}x$$
   $$= -e^{-x}(x^2+1) - 2xe^{-x} - 2e^{-x} + c = -e^{-x}(x^2-2x+3) + c$$

Example (Integration by parts II)

1. $$\int \ln x \,\mathrm{d}x = \left| \begin{array}{ll} u = \ln x & v' = 1 \\ u' = \frac{1}{x} & v = x \end{array} \right| = x\ln x - \int \frac{1}{x}\cdot x \,\mathrm{d}x$$
$$= x\ln x - \int \mathrm{d}x = x\ln x - x + c$$

2. $$\int x \operatorname{arctg} x \,\mathrm{d}x = \left| \begin{array}{ll} u = \operatorname{arctg} x & v' = x \\ u' = \frac{1}{x^2+1} & v = \frac{x^2}{2} \end{array} \right| = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int \frac{x^2}{x^2+1}\,\mathrm{d}x$$
$$= \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}\int \left(1 - \frac{1}{x^2+1}\right)\mathrm{d}x = \frac{x^2}{2}\operatorname{arctg} x - \frac{1}{2}(x - \operatorname{arctg} x) + c$$

Using the computer algebra systems

Wolfram Alpha:

`http://www.wolframalpha.com/`

Mathematical Assistant on Web (MAW):

`wood.mendelu.cz/math/maw-html/index.php?lang=en&form=integral`

Example

Using the Wolfram Alpha find the integral

$$\int \ln x \,\mathrm{d}x.$$

Solution:

`integrate ln x dx`