european social fund in the czech republic
EUROPEAN UNION
MINISTRY OF EDUCATION, YOUTH AND SPORTS
OP Education for Competitiveness
INVESTMENTS IN EDUCATION DEVELOPMENT

Execrices

Mathematics

FRDIS

Vytvořeno s podporou projektu Průřezová inovace studijních programů Lesnické a dřevařské fakulty MENDELU v Brně (LDF) s ohledem na disciplíny společného základu `http://akademie.ldf.mendelu.cz/cz` (reg. č. CZ.1.07/2.2.00/28.0021) za přispění finančních prostředků EU a státního rozpočtu České republiky.

Simona Fišnarová MENDELU

1 Derivatives

Find the derivative of the functions:

1. $y = \sqrt{x} + \ln \sin \operatorname{tg} x$
2. $y = \dfrac{1}{x^3} + \ln \cos \dfrac{x}{2}$
3. $y = \dfrac{1}{\sqrt{x}} + \operatorname{arctg} e^{3x}$
4. $y = x^3 \ln x + \sin x^2$
5. $y = \dfrac{\ln x}{x^2} + (\sin 3x + 4)^5$
6. $y = e^x(x^2 + 3) + \sin(\ln x^2)$
7. $y = x^2 e^x + \ln(\cos 3x)$
8. $y = \dfrac{\sin x}{x^2} + e^{\sin 2x}$
9. $y = e^x(x^2 + 3x) + \cos(\ln x^2)$
10. $y = x^3 \cos x + \ln(\sin x^3)$
11. $y = \dfrac{x^3}{x^2 + 1} + \sin(e^{x^3})$
12. $y = x^2 \ln x + \sin^2(e^{2x})$
13. $y = \dfrac{\sin x}{x} + \ln(\cos^3 x)$
14. $y = \dfrac{x^3}{x + 2} + \cos(e^{x^2})$
15. $y = \sqrt{x} \ln x + e^{\sin x^3}$
16. $y = \dfrac{e^x}{x^3} + (x^2 + \ln 3x)^4$
17. $y = e^{2x}(x^2 + 1) + \ln^2 x$
18. $y = \sqrt{\dfrac{x}{x + 2}}$
19. $y = \ln\left(\dfrac{x - 1}{x + 1}\right)$

2 Investigation of functions

2.1 Polynomials

Investigate the following polynomials:

(a) Find the domain, find the parts above/below x-axis.

(b) Monotonicity.

(c) Concavity.

(d) Sketch the graph.

1. $y = 6x^3 - 3x^6$.
2. $y = x^5 - 5x^4$.
3. $y = 3x - x^3$.
4. $y = x^4 + 2x^3$.
5. $y = 4x^3 - x^4$.
6. $y = x^3 - 2x^2 + x$.
7. $y = x^3 - 4x^2 + 4x$.
8. $y = x^3 - 6x^2 + 9x$.
9. $y = 2x^3 - 9x^2 + 12x$.

2.2 Rational functions

Investigate the following functions:

(a) Find the domain, find the parts above/below x-axis.

(b) Monotonicity.

(c) Concavity.

(d) Asymptotes.

(e) Sketch the graph.

1. $y = \dfrac{x^2}{x-2}$

   Hint: $y' = \dfrac{x^2 - 4x}{(x-2)^2}$, $y'' = \dfrac{8}{(x-2)^3}$.

2. $y = \dfrac{x^2}{x+1}$

   Hint: $y' = \dfrac{x^2 + 2x}{(x+1)^2}$, $y'' = \dfrac{2}{(x+1)^3}$.

3. $y = \dfrac{x}{(x-2)^2}$

   Hint: $y' = \dfrac{-x-2}{(x-2)^3}$, $y'' = \dfrac{2x+8}{(x-2)^4}$.

4. $y = \dfrac{x}{(x+3)^2}$

   Hint: $y' = \dfrac{3-x}{(x+3)^3}$, $y'' = \dfrac{2x-12}{(x+3)^4}$.

5. $y = \dfrac{x^2}{(x+1)^2}$.

   Hint: $y' = \dfrac{2x}{(x+1)^3}$, $y'' = \dfrac{2-4x}{(x+1)^4}$.

6. $y = \dfrac{x}{x^2+1}$.

   Hint: $y' = \dfrac{1-x^2}{(x^2+1)^2}$, $y'' = \dfrac{2x^3-6x}{(x^2+1)^3}$.

7. $y = \dfrac{x}{(x-1)^2}$.

   Hint: $y' = \dfrac{-x-1}{(x-1)^3}$, $y'' = \dfrac{2x+4}{(x-1)^4}$.

8. $y = \dfrac{x-2}{(x-1)^2}$.

   Hint: $y' = \dfrac{3-x}{(x-1)^3}$, $y'' = \dfrac{2x-8}{(x-1)^4}$.

9. $y = x + \dfrac{1}{x+1}$.

   Hint: $y' = \dfrac{x^2+2x}{(x+1)^2}$, $y'' = \dfrac{2}{(x+1)^3}$.

10. $y = \dfrac{x^2-1}{x^3}$.

    Hint: $y' = \dfrac{3-x^2}{x^4}$, $y'' = \dfrac{2x^2-12}{x^5}$.

3 Integrals

3.1 Basic integrals

1. $\int x^2(x+3)\,\mathrm{d}x$
2. $\int \sqrt{x}\,\mathrm{d}x$
3. $\int x^2\sqrt{x}\,\mathrm{d}x$
4. $\int \frac{1}{x^2}\,\mathrm{d}x$
5. $\int \frac{1}{\sqrt{x}}\,\mathrm{d}x$
6. $\int \frac{1}{x^4}\,\mathrm{d}x$
7. $\int \frac{2}{x^3}\,\mathrm{d}x$
8. $\int \sqrt[3]{x}\,\mathrm{d}x$
9. $\int \frac{x+1}{x^3}\,\mathrm{d}x$
10. $\int \frac{x^2+x+3}{x^2}\,\mathrm{d}x$

3.2 Basic substitutions

1. $\int e^{-3x}\,\mathrm{d}x$
2. $\int \cos 3x\,\mathrm{d}x$
3. $\int e^{-x}\,\mathrm{d}x$
4. $\int \sin 5x\,\mathrm{d}x$
5. $\int e^{4x}\,\mathrm{d}x$
6. $\int \cos\frac{x}{2}\,\mathrm{d}x$
7. $\int \sin\frac{x}{3}\,\mathrm{d}x$
8. $\int (3x-5)^5\,\mathrm{d}x$
9. $\int \frac{1}{2x-3}\,\mathrm{d}x$
10. $\int \frac{1}{(2x+5)^3}\,\mathrm{d}x$
11. $\int \frac{x}{x^2+2}\,\mathrm{d}x$
12. $\int \operatorname{tg} x\,\mathrm{d}x$
13. $\int \operatorname{cotg} x\,\mathrm{d}x$
14. $\int \frac{3x}{x^2+5}\,\mathrm{d}x$

3.3 Substitutions

1. $\int x\sin x^2\,\mathrm{d}x$
2. $\int x\sin(x^2+3)\,\mathrm{d}x$
3. $\int xe^{x^2}\,\mathrm{d}x$
4. $\int x\cos x^2\,\mathrm{d}x$
5. $\int x^2e^{x^3+2}\,\mathrm{d}x$
6. $\int \sin^3 x\cos x\,\mathrm{d}x$
7. $\int \frac{\cos x}{\sin^2 x}\,\mathrm{d}x$
8. $\int \frac{\sin x}{\cos^4 x}\,\mathrm{d}x$
9. $\int \frac{\sin x}{\sin^2 x+5}\cos x\,\mathrm{d}x$
10. $\int \frac{x}{\sqrt{x-2}}\,\mathrm{d}x$
11. $\int \frac{\cos x}{\cos^2 x+7}\sin x\,\mathrm{d}x$
12. $\int \frac{x}{\sqrt{x+3}}\,\mathrm{d}x$
13. $\int \frac{3}{x\sqrt{x-1}}\,\mathrm{d}x$
14. $\int \frac{1}{x+\sqrt{x}}\,\mathrm{d}x$

3.4 By parts

1. $\int x \cos x \, \mathrm{d}x$
2. $\int x^2 e^x \, \mathrm{d}x$
3. $\int \ln x \, \mathrm{d}x$
4. $\int x e^x \, \mathrm{d}x$
5. $\int x \ln x \, \mathrm{d}x$
6. $\int x^2 \sin x \, \mathrm{d}x$
7. $\int x^2 \ln x \, \mathrm{d}x$
8. $\int x \cos 3x \, \mathrm{d}x$
9. $\int \operatorname{arctg} x \, \mathrm{d}x$
10. $\int x \operatorname{arctg} x \, \mathrm{d}x$
11. $\int x e^{-x} \, \mathrm{d}x$
12. $\int x \sin 2x \, \mathrm{d}x$

3.5 Definite integrals

1. $\int_0^1 (x^2 + x + 2) \, \mathrm{d}x$
2. $\int_0^2 (x^2 + 1) \, \mathrm{d}x$
3. $\int_1^2 (2x + 1) \, \mathrm{d}x$
4. $\int_1^2 (x^3 - 2x + 1) \, \mathrm{d}x$
5. $\int_0^1 (x^3 - 3x^2 + 1) \, \mathrm{d}x$
6. $\int_1^2 \frac{x^3 + 2x + 3}{x^2} \, \mathrm{d}x$
7. $\int_0^\pi x \cos x \, \mathrm{d}x$
8. $\int_0^1 x e^{x^2+1} \, \mathrm{d}x$
9. $\int_0^3 \frac{x}{\sqrt{x+1}} \, \mathrm{d}x$

4 Vectors, matrices, determinants

4.1 Operations with matrices

1. Let
$$A = \begin{pmatrix} 3 & 0 & 3 \\ 0 & -1 & 2 \\ 3 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 \\ 0 & 3 \\ 2 & 3 \end{pmatrix}.$$
Calculate $(A - 2I)^T \cdot B$, where I is the identity matrix .

2. Let
$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & 2 \\ 0 & 2 \\ 2 & 1 \end{pmatrix}.$$
Calculate $(A^T + I) \cdot B$, where I is the identity matrix.

3. Let
$$A = \begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 0 \\ 3 & 1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix},$$
Calculate $(A - B)^2$, where I is the identity matrix.

4. Let
$$A = \begin{pmatrix} 1 & 5 & 2 \\ 2 & 0 & 1 \\ 3 & 2 & 0 \end{pmatrix}.$$
Calculate A^2.

5. Let
$$A = \begin{pmatrix} 1 & 1 & 3 \\ 2 & 2 & 1 \\ 2 & 2 & 0 \end{pmatrix}.$$
Calculate $(A^T - I)A$, where I is the identity matrix.

4.2 Determiannts, inverse matrix, linear dependence/independence of vectors

1. Let
$$A = \begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$
   (a) Evaluate the determinant of A.
   (b) Using the value of $\det A$ answer the following questions:
      (i) Are the rows of A linearly dependent or independent?
      (ii) Is $\text{rank}\,(A) > 3$, $\text{rank}\,(A) < 3$ or $\text{rank}\,(A) = 3$?
      (iii) Does the inverse matrix A^{-1} exist? If A^{-1} exists, find it.

2. Let
$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ 3 & 2 & 4 \end{pmatrix}.$$
   (a) Evaluate the determinant of A.
   (b) Using the value of $\det A$ answer the following questions:
      (i) Are the rows of A linearly dependent or independent?
      (ii) Is $\text{rank}\,(A) > 3$, $\text{rank}\,(A) < 3$ or $\text{rank}\,(A) = 3$?
      (iii) Does the inverse matrix A^{-1} exist? If A^{-1} exists, find it.

3. Let
$$\begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 0 & 1 & 1 \end{pmatrix}.$$
   (a) Evaluate the determinant of A.
   (b) Using the value of $\det A$ answer the following questions:
      (i) Are the rows of A linearly dependent or independent?
      (ii) Is $\text{rank}\,(A) > 3$, $\text{rank}\,(A) < 3$ or $\text{rank}\,(A) = 3$?
      (iii) Does the inverse matrix A^{-1} exist? If A^{-1} exists, find it.

4. Let
$$A = \begin{pmatrix} 1 & 1 & 2 \\ 0 & -1 & 0 \\ -1 & -2 & -1 \end{pmatrix}.$$
   (a) Evaluate the determinant of A.
   (b) Using the value of $\det A$ answer the following questions:
      (i) Are the rows of A linearly dependent or independent?
      (ii) Is $\text{rank}\,(A) > 3$, $\text{rank}\,(A) < 3$ or $\text{rank}\,(A) = 3$?
      (iii) Does the inverse matrix A^{-1} exist? If A^{-1} exists, find it.

5. Let
$$A = \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & 2 \\ 2 & 1 & 5 \end{pmatrix}.$$
   (a) Evaluate the determinant of A.
   (b) Using the value of $\det A$ answer the following questions:
      (i) Are the rows of A linearly dependent or independent?
      (ii) Is $\text{rank}\,(A) > 3$, $\text{rank}\,(A) < 3$ or $\text{rank}\,(A) = 3$?
      (iii) Does the inverse matrix A^{-1} exist? If A^{-1} exists, find it.

6. Evaluate the determinants:
$$\begin{vmatrix} 3 & 2 & 3 & 1 \\ 0 & 0 & 2 & 0 \\ 3 & 1 & 2 & 2 \\ 0 & 3 & 2 & 1 \end{vmatrix}, \quad \begin{vmatrix} 3 & 3 & 1 & 0 \\ 5 & 3 & 2 & 3 \\ 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{vmatrix}, \quad \begin{vmatrix} 5 & 3 & 2 & 3 \\ 2 & 0 & 0 & 0 \\ 3 & 3 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{vmatrix}, \quad \begin{vmatrix} 1 & 5 & 1 & 0 \\ 2 & 0 & 3 & 3 \\ 0 & 3 & 0 & 0 \\ 3 & -3 & 1 & -2 \end{vmatrix}.$$

7. Are the following vectors liearly dependent or independent?

(a) $\vec{a} = (1, 2, 1, 0),\ \vec{b} = (1, 2, -1, 1),\ \vec{c} = (0, 1, 2, 1),\ \vec{d} = (1, 1, 0, 1)$

(b) $\vec{a} = (1, 2, 1, 0),\ \vec{b} = (1, 0, -1, 1),\ \vec{c} = (1, 1, 2, 1),\ \vec{d} = (2, 1, 1, 2)$

(c) $\vec{a} = (1, 3, 1, 0),\ \vec{b} = (1, -1, 0, 1),\ \vec{c} = (1, 1, 2, 1),\ \vec{d} = (1, 1, 1, 2)$

5 Systems of linear equations

Solve the following systems using the Gauss method.

(a) Find the rank of the coeffcient and of the augmented matrix and determine how many solutions the system has.

(b) Find the solution of the system (if exists any).

1.

$$\begin{aligned} 8x_1 + 6x_2 - x_3 + 3x_4 &= -9 \\ 2x_1 + 2x_2 - x_3 + 5x_4 &= -13 \\ x_1 + 2x_2 - 2x_3 + 11x_4 &= -28 \\ 2x_2 - 3x_3 + 17x_4 &= -43. \end{aligned}$$

2.

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= -2 \\ 2x_1 + x_2 - x_3 + 2x_4 &= 2 \\ 3x_1 + 2x_2 - 2x_3 + 3x_4 &= 1 \\ x_2 - 3x_3 + 2x_4 &= -3. \end{aligned}$$

3.

$$\begin{aligned} x_1 + 2x_2 - x_4 &= -2 \\ 2x_1 + 3x_2 + x_3 - 5x_4 &= 1 \\ x_1 + x_2 + x_3 - 4x_4 &= 3 \\ x_2 - x_3 + 2x_4 &= 0. \end{aligned}$$

4.

$$\begin{aligned} x_1 + x_2 - 2x_3 + 3x_4 &= 0 \\ 3x_1 + 2x_2 + 3x_3 - 4x_4 &= -4 \\ -3x_1 - 2x_2 - 3x_3 + 3x_4 &= 4 \\ -7x_1 - 6x_2 + 5x_3 - 8x_4 &= 4. \end{aligned}$$

5.

$$\begin{aligned} x_1 + x_2 + 3x_3 - x_4 &= 2 \\ 2x_1 + x_2 + 5x_3 - 2x_4 &= 0 \\ 2x_1 - x_2 + 3x_3 - 2x_4 &= -8 \\ 3x_1 + 2x_2 + 8x_3 - 3x_4 &= 2. \end{aligned}$$

6.

$$\begin{aligned} x_1 + 3x_2 - 2x_3 + x_4 &= 0 \\ 2x_1 + 5x_2 - 3x_3 + 3x_4 &= 0 \\ x_1 + 2x_3 - 2x_4 &= 9 \\ 2x_1 - x_2 + 4x_3 + 9x_4 &= 3. \end{aligned}$$

7.

$$\begin{aligned} x_1 + 3x_2 + 2x_3 - 4x_4 &= -4 \\ x_2 + x_3 - 3x_4 &= -3 \\ -x_1 + 2x_2 + x_3 - x_4 &= -1 \\ 5x_1 + 2x_2 + 4x_4 &= 4. \end{aligned}$$

8.

$$\begin{aligned} x_1 + 2x_2 - 5x_3 + x_4 &= -2 \\ x_2 + 3x_3 - 4x_4 &= 1 \\ -x_1 + 2x_2 - x_3 + x_4 &= 6 \\ 3x_1 + x_2 - 4x_3 + 6x_4 &= -2. \end{aligned}$$

9.

$$\begin{aligned} x_1 + x_2 - x_3 + x_4 &= 0 \\ 2x_1 + 3x_2 + x_3 + x_4 &= 6 \\ 4x_1 + 5x_2 - x_3 + 3x_4 &= 6 \\ 3x_1 + 4x_2 - 6x_3 + 2x_4 &= -6. \end{aligned}$$

10.

$$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 &= 1 \\ x_1 - 2x_2 - x_3 + 2x_4 &= 1 \\ 2x_1 + 3x_3 + x_4 &= 2 \\ x_1 + x_2 + 3x_3 &= 1. \end{aligned}$$

11.

$$\begin{aligned} x_1 + x_2 + 2x_4 &= 0 \\ x_1 + x_3 + x_4 &= 2 \\ 2x_1 + x_2 + x_3 + 3x_4 &= 3 \\ x_2 - 2x_3 + 3x_4 &= 1. \end{aligned}$$

12.

$$\begin{aligned} x_1 + x_2 + 5x_4 &= 1 \\ x_1 + x_3 + 2x_4 &= 1 \\ x_1 - 3x_2 + 4x_3 - 7x_4 &= 1 \\ x_2 - x_3 + 3x_4 &= 0. \end{aligned}$$

Solve the following systems using

(a) the Cramer rule,

(b) the inverse of the coefficient matrix.

1.

$$\begin{aligned} 2x + 3y &= 1 \\ 4x + 7y &= 3 \end{aligned}$$

2.

$$\begin{aligned} 1x + 2y &= 3 \\ 3x + 5y &= 2 \end{aligned}$$